# A new measure of robust stablity for linear ordinary impulsive differential equations

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## 1 Introuction

Impulsive differential equations are frequently employed as models of biological, chemical, and physical systems, among others. The property of these equations that makes them attractive in applications is the impulse effect, which allows for the inclusion of fast dynamics that might otherwise complicate the analysis of the model, were they to be included in a continuous, as opposed to discrete, manner. Most monographs on impulsive differential equations — for example, [1, 5, 6] — state that these equations are reasonable approximations of continuous models with perturbations, if the perturbations themselves occur quickly, relative to the overall dynamics. In practice, this is often taken as an assumption of the model in question. Methods to determine how quickly these perturbations must occur for the model to be a good "fit" to the associated continuous models have yet to be seen in the literature.

In these proceedings, we introduce a quantity that we call the *time-scale tolerance*, denote $\mathscr{E}_t$, for a linear, periodic impulsive differential equation that is asymptotically stable. This quantity has the property, that, if the vector of durations of impulse effect, $a$, satisfies $||a|| < \mathscr{E}_t$, then both the impulsive model and a family of continuous impulse extension equations (a specific functional differential equation) to which it is related, will all be asymptotically stable.

We review linear impulse extension equations, which were first introduced in [2, 3, 4], state theorems that describe the convergence of their solutions to the associated impulsive solutions, and introduce all the machinery necessary in the development of the time-scale tolerance, stating theoretical results on its existence and how it can be computed in practice. We conclude with two illustrative examples and a

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discussion of the limitations of the present techniques, and how they can be extended to accomodate a larger class of problems.

## 2 Linear impulse extension equations

To begin, we introduce some notation that will be present throughout this chapter. If $x = \{x_k : k \in \mathbb{Z}\}$ is a real-valued sequence, we denote $\Delta x_k = x_{k+1} - x_k$. The $k$th element of a real-valued sequence $x$ will always be denoted $x_k$, and we may abuse notation and identify the sequence $x$ with the symbol $x_k$. Indexed families of sequences, such as, $\{x^j : j \in U\}$, will always have their index appear in the exponent. In this context, $x_k^j$ denotes the $k$th element of sequence $j$ from the family $U$. Finally, our sequences will usually be bi-infinite; that is, indexed by the integers. The symbol $||\cdot||$ will denote a (fixed) Euclidean norm, whenever there is no ambiguity, and if $A$ is a set, its closure will be denoted $\overline{A}$.

Consider a linear, impulsive differential equation with impulses at fixed times

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), & t &\neq \tau_k \\ \Delta x &= B_k x + h_k, & t &= \tau_k. \end{aligned} \tag{1}$$

with $t \in \mathbb{R}$, phase space $\Omega \subset \mathbb{R}^n$, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times n}$, $h_k \in \mathbb{R}^n$, and sequence of impulses $\tau_k$ for $k \in \mathbb{Z}$. We assume that $A$ and $g$ are locally integrable, and that the sequence of impulse times, $\tau_k$, is monotone increasing and unbounded.

An *impulse extension equation for* (1), as described in Church and Smith? [3, 4] and Church [2], is defined as follows.

**Definition 1.** Consider a linear impulsive differential equation (1).

- A *step sequence over* $\tau_k$ is sequence of positive real numbers $a = \{a_k : k \in \mathbb{Z}\}$ such that $a_k \in (0, \Delta\tau_k)$ for all $k \in \mathbb{Z}$. We denote $S_j = S_j(a) \equiv [\tau_j, \tau_j + a_j)$ and $S = S(a) \equiv \bigcup_{j\in\mathbb{Z}} S_j$. The set of all step sequences will be denoted $S^*$, and is defined by
$$S^* := \{a : \mathbb{Z} \to \mathbb{R}\ ,\ a_k \in (0, \Delta\tau_k)\}.$$
- The pair $(\varphi_k^B, \varphi_k^h)$, with sequences of functions $\varphi_k^B : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $\varphi_k^h : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}^n$, is a *family of impulse extension for* (1) if for all $a \in S^*$ and all $k \in \mathbb{Z}$, the functions $\varphi_k^B(\cdot, a_k)$ and $\varphi_k^h(\cdot, a_k)$ are integrable on $S_k(a)$ and satisfy the equalities
$$\int_{S_k(a)} \varphi_k^B(t, a_k)dt = B_k, \qquad \int_{S_k(a)} \varphi_k^h(t, a_k)dt = h_k. \tag{2}$$
- Given a step sequence $a \in S^*$ and a family of impulse extensions $\varphi = (\varphi_k^B, \varphi_k^h)$ for (1), the *impulse extension equation associated to* (1) *and induced by* $(\varphi, a)$ is the (functional) differential equation

$$\frac{dx}{dt} = \begin{cases} A(t)x + g(t), & t \notin S(a), \\ A(t)x + g(t) + \varphi_k^B(t, a_k)x(\tau_k) + \varphi_k^h(t, a_k), & t \in S_k(a). \end{cases} \tag{3}$$

**Definition 2.** Let a family of impulse extensions, $\varphi = (\varphi_k^B, \varphi_k^h)$, and a step sequence $a \in S^*$ be given. A function $y : I \to \mathbb{R}^n$ defined on an interval $I \subset \mathbb{R}$ is a *classical solution* of the impulse extension equation (3) induced by $(\varphi, a)$, if $y$ is continuous, the sets $I \cap S_k(a)$ are either empty or contain $\tau_k$, and $y$ satisfies the differential equation (3) almost everywhere on $I$. Given an *initial condition*

$$x(t_0) = x_0, \tag{4}$$

with $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, the function $y(t)$ is a solution of the *initial-value problem* (3)-(4) if, in addition, $y(t_0) = x_0$.

*Remark 1.* Definition 1 and Definition 2 can be readily modified to accomodate nonlinear ordinary impulsive differential equations; see Church and Smith? [4] and Church [2].

**Definition 3.** The *predictable set* of the impulse extension equation (3) induced by $(\varphi, a)$ is

$$\mathscr{P} = \mathbb{R} \setminus \left\{ t \in \overline{S(a)} : \det\left( I + \int_{\max_{\tau_k}\{\tau_k \leq t\}}^{t} X^{-1}(s, \tau_k)\varphi_k^B(s, a_k)ds \right) = 0 \right\},$$

where $X(t,s)$ is the Cauchy matrix of the homogeneous system $z' = A(t)z$.

The following theorem states the mode in which solutions of the impulse extension equation, (3), converge to those of the impulsive differential equation, (1).

**Theorem 1.** *Suppose* $\det(I + B_k) \neq 0$ *for all* $k \in \mathbb{Z}$*, and let* $\varphi = (\varphi_k^B, \varphi_k^h)$ *be a given family of impulse extensions for* (1)*. There exists a positive sequence of real numbers* $\sigma_k$*, depending only on* $A(t)$ *and sequence of impulse times* $\tau_k$*, with the following property. Suppose, for* $\xi \in \{B, h\}$ *and each* $k \in \mathbb{Z}$*, there exists* $w_k^\xi : [\tau_k, \tau_{k+1}] \times [0, \Delta\tau_k) \to \mathbb{R}$ *that is continuous and vanishing at* $(\tau_k, 0)$*, for which*

$$\varphi_k^\xi(t,s) - \frac{1}{s}\xi_k = O\left( w_k^\xi(t,s)\frac{1}{e^{\sigma_k s} - 1} \right) \tag{5}$$

*for* $t \in [\tau_k, \tau_k + s)$ *as* $s \to 0$*. The following are true:*

- *For all* $t_0 \in \mathbb{R}$*, there exists* $\delta > 0$*, such that, for* $a \in S^*$ *with* $||a||_\infty < \delta$ *and all* $x_0 \in \mathbb{R}^n$*, the impulse extension equation* (3) *induced by* $(\varphi, a)$ *possesses a unique classical solution,* $x(t; a)$*, satisfying the initial condition* $x(t_0) = x_0$*.*
- *The function* $x(t; a)$ *converges pointwise to* $x(t; 0)$*, the solution the initial value problem* $x(t_0) = x_0$ *for the impulsive differential equation,* (1)*, as* $a \to 0$*.*
- *If* $N \subset \mathbb{R}$ *is bounded and no strictly decreasing sequence in* $N$ *has an impulse time* $\tau_k$ *as its limit, the above convergence is uniform for* $t \in N$*.*

*Proof (Outline).* The existence of the sequence $\sigma_k$ follows from the Generalized Gronwall's inequality: we have $||X(t;\tau_k)|| \leq e^{\sigma_k(t-\tau_k)}$ for $t \in [\tau_k, \tau_{k+1}]$, where $\sigma_k = \int_{\tau_k}^{\tau_{k+1}} ||A(s)||ds$, and $X(t;s)$ is the Cauchy matrix of $x' = A(t)x$. When $||a||_\infty < t_0 - \max\{\tau_k : \tau_k < t_0\}$, the solution of (3)–(4) induced by $(\varphi, a)$ satisfying $x(t_0;a) = x_0$, can be written as

$$x(t;a) = U(t;a)x_0 + x_p(t;a),$$

for a matrix function $t \mapsto U(t;a)$ satisfying $U(t_0;a) = I$, and $x_p(t_0;a) = 0$ (this follows by Proposition 4.2 of [4]). If $U(t;0)$ denotes the fundamental matrix solution of the homogeneous equation associated to (1), one can show that the inequality

$$\begin{aligned} ||U(t,a) - U(t,0)|| \leq ||X(t;\tau_k)|| \cdot \Bigg|\Bigg| \Bigg( L_a(t;\tau_k) \prod_{r=k-1}^{0} X(\tau_{r+1};\tau_r) L_a(\tau_r + a_r;\tau_r) \Bigg) \\ - \ (I + B_k) \Bigg( \prod_{r=k-1}^{0} X(\tau_{r+1};\tau_r)(I + B_r) \Bigg) \Bigg|\Bigg| \end{aligned}$$

holds, where we have assumed $t_0 = \tau_0$ for ease of presentation (other cases follow by similar reasoning, by results from [4]), and

$$L_a(t;\tau_k) = I + \int_{\tau_k}^{\min\{t,\tau_k + a_k\}} X^{-1}(s;\tau_k)\varphi_k^B(s,a_k)ds.$$

It can be shown that the right-hand side of the upper bound converges to zero as $a \to 0$ pointwise, as $||a||_\infty \to 0$ (see the proof of Theorem 3.5.5. from [2] for the main idea; condition (5) is needed). A similar inequality holds for $||x_p(t;a) - x_p(t;0)||$, where $x_p(t;0)$ is the solution of (1) satisfying $x_p(t_0;0) = 0$, and the convergence result holds for that piece of the solution as well. For uniform convergence, it suffices to consider $N$ to be a finite union of closed intervals with $x_n \downarrow x \in N \Rightarrow x \notin \{\tau_k\}$.

The hypotheses of Theorem 1 are simplified if the equations (1) and (3) are periodic.

**Definition 4.** The linear impulsive differential equation (1) is *$T$-periodic with $c$ impulses per period* if $A(t+T) = A(t)$ and $g(t+T) = g(t)$ for all $t \in \mathbb{R}$, and $\tau_{k+c} = \tau_k + T$, $B_{k+c} = B_k$ and $h_{k+c} = h_k$ for all $k \in \mathbb{Z}$. The step sequence $a \in S^*$ is *$c$-periodic*, and we write $a \in S_c^*$, if $a_{k+c} = a_k$ for all $k \in \mathbb{Z}$. The family of impulse extensions $\varphi = (\varphi_k^B, \varphi_k^h)$ is *$(T,c)$-periodic* if $\varphi_{k+c}^\xi(t+T,s) = \varphi_k^\xi(t,s)$ for all $t \in \mathbb{R}$, all $k \in \mathbb{Z}$, all $s \in (0, \Delta\tau_k)$ and $\xi \in \{B,h\}$.

**Corollary 1.** *Suppose the impulsive differential equation* (1) *is $T$-periodic with $c$ impulses per period. Let $\varphi = (\varphi_k^B, \varphi_k^h)$ be a $(T,c)$-periodic family of impulse extensions for* (1)*. Suppose* $\det(I + B_k) \neq 0$ *for $k = 0, \ldots, c-1$. Let the impulsive differential equation* (1) *have a fundamental matrix $X(t)$ with Floquet decomposition $X(t) = P(t)e^{\Lambda t}$ satisfying $X(\tau_0) = I$. The conclusions of Theorem 1 hold for step sequences $a \in S_c^*$, with $\sigma_k \equiv ||\Lambda||$.*

The proof of the above corollary is omitted, since it is simple to prove using Theorem 1. For periodic impulse extension equations, we have an asymptotic Floquet theorem. A proof is available in [2], where it is listed as Theorem 3.6.16.

**Theorem 2.** *Suppose the impulsive differential equation* (1) *is $T$-periodic with $c$ impulses per period. Let $\varphi = (\varphi_k^B, \varphi_k^h)$ be a $(T,c)$-periodic family of impulse extensions for* (1)*. Let* $\det(I+B_k) \neq 0$ *for $k = 0,\dots,c-1$. Then, under the hypotheses of Corollary 1 on the asymptotic criterion* (5)*, there exists $\delta > 0$ such that, if $a \in S_c^*$ satisfies $||a|| < \delta$, any solution $x(t)$ of the* homogeneous impulse extension equation induced by $(\varphi, a)$,

$$\frac{dx}{dt} = \begin{cases} A(t)x, & t \notin S(a) \\ A(t)x + \varphi_k^B(t,a_k)x(\tau_k), & t \in S_k(a). \end{cases} \tag{6}$$

*can be written as a product,*

$$x(t) = U_a(t)x_0 = P_a(t)e^{\Lambda_a t}x_0, \tag{7}$$

*for some $x_0 \in \mathbb{R}^n$, $T$-periodic matrix $P_a(t)$, and nonsingular $\Lambda_a$. $U_a$ can be normalized so that $U_a(\tau_0) = I$, and in this case, we have $\Lambda_a \to \Lambda_0$ as $a \to 0$, where $\Lambda_0$ is the matrix appearing in the Floquet decomposition, $X(t) = U_0(t) = P_0(t)e^{\Lambda_0 t}$, of the homogeneous equation associated to the periodic impulsive differential equation,* (1)*, with $U_0(\tau_0) = I$.*

The stability of the periodic linear impulse extension equation induced by some $(\varphi, a)$ is determined by the spectrum of $\Lambda_a$, just as with ordinary and impulsive differential equations. The main difference is that stability (and uniform stability) only holds for initial conditions in particular subsets of the predictable set, $\mathscr{P}$, and such restrictions are in fact, optimal. For details, see [4].

## 3 The time-scale tolerance for linear, periodic impulsive differential equations

Stability of (3) is completely determined by the associated homogeneous equation (6); see [4]. As such, our investigation will now shift to homogeneous, $(T,c)$-periodic impulsive differential equations,

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, & t &\neq \tau_k \\ \Delta x &= B_k x, & t &= \tau_k. \end{aligned} \tag{8}$$

and impulse extension equations for (8), induced by $(\varphi, a) = (\varphi^B, a)$, with $a \in S_c^*$,

$$\frac{dx}{dt} = \begin{cases} A(t)x, & t \notin S(a) \\ A(t)x + \varphi_k^B(t,a_k)x(\tau_k), & t \in S_k(a). \end{cases} \tag{9}$$

From here onward, $M_0$ will denote the monodromy matrix for (8) satisfying $M_0 = X(\tau_0 + T, \tau_0)$, where $X(t,s)$ is the Cauchy matrix for (8). We assume $\rho M_0 < 1$ from here onward. We will comment in Section 3.3 on what can be done if $\rho M_0 \geq 1$.

**Definition 5.** Consider a periodic homogeneous impulsive differential equation, (8). Let $\sigma = \{\sigma_k\}$ be a $c$-element sequence of positive real numbers and $w = \{w_k\}$ be a $c$-element sequence of functions $w_k : [\tau_k, \tau_{k+1}] \times \overline{S_c^*} \to \mathbb{R}_+$ that are continuous and vanishing at $(\tau_k, 0)$ and such that $w_k(\cdot, a)$ is integrable on $S_k(a)$. A family of periodic impulse extensions, $\varphi = \{\varphi_k\}$, is *uniformly exponentially* $(\sigma, w)$*-regulated in the mean* or simply $(\sigma, w)$*-regulated* if the inequality

$$\left|\left|\varphi_k(s,a) - \frac{1}{a_k}B_k\right|\right| \leq \frac{w_k(s,a)}{e^{\sigma_k a_k} - 1} \tag{10}$$

is satisfied, for all $s \in S_k(a)$ and $k = 0, \ldots, c-1$. A pair $(\sigma, w)$ that satisfies the above criteria will be referred to as a *uniform exponential regulator*, or simply *exponential regulator*. If $\varphi$ is uniformly $(\sigma, w)$-regulated, we will write $\varphi \in (\sigma, w)$.

**Definition 6.** If $R = (\sigma, w)$ is an exponential regulator, the $(R, a)$*-pseudospectral radius of* (8)*,* denoted $\rho(R, a)$, is defined by

$$\rho(R,a) = \sup_{\varphi \in R} \rho M(\varphi, a), \tag{11}$$

and $M(\varphi, a)$ denotes the monodromy matrix of the impulse extension equation for (8) induced by $(\varphi, a)$.

**Definition 7.** Suppose (8) is asymptotically stable. Let $R$ be an exponential regulator. The $R$*-stable set*, denoted $\mathscr{E}_s(R)$, is defined as follows.

$$\mathscr{E}_s(R) = \{a \in S_c^* : \forall \varphi \in (\sigma, w),\, \rho M(\varphi, a) < 1\} \tag{12}$$

The $R$*-time-scale tolerance* is the number

$$\mathscr{E}_t(R) = \sup\{\varepsilon : \exists a \in \mathscr{E}_s(R), ||a|| = \varepsilon, B_\varepsilon(0) \cap S_c^* \subseteq \mathscr{E}_s(R)\}. \tag{13}$$

The time-scale tolerance is defined precisely so that we have the following elementary property, whose proof we omit.

**Proposition 1.** *Given an exponential regulator* $R = (\sigma, w)$*, the time-scale tolerance behaves as a robust stability threshold for the impulsive system* (1)*; if* $||a|| < \mathscr{E}_t(R)$*, then* $\rho(R, a) < 1$*. In other words, systems* (8) *and the impulse extension equation* (6) *induced by* $(\varphi, a)$ *are both stable, for all* $\varphi \in R$*.*

**Theorem 3.** *Suppose* $\sigma = ||\Lambda||$*, as in Corollary 1. If* $R_\sigma = (\sigma, w)$ *is an exponential regulator and* (8) *is asymptotically stable, then* $\mathscr{E}_t(R_\sigma)$ *is nonzero and the map* $a \mapsto \rho(R_\sigma, a)$ *satisfies*

$$\lim_{a \to 0} \rho(R_\sigma, a) = \rho M_0,$$

*where the limit is for* $a \in S_c^*$*.*

*Proof (Outline).* The monodromy matrix for (6) induced by $(\varphi, a)$ for $\varphi \in R_\sigma$ can be written as

$$M(\varphi,a)=\prod_{k=c-1}^{0}X(\tau_{k+1};\tau_k)\left(\int_{S_k(a)}X^{-1}(s;\tau_k)\varepsilon_k(s,a)ds+\frac{1}{a_k}\int_{S_k(a)}I+X^{-1}(s;\tau_k)B_k ds\right),$$

with $\varepsilon_k(s,a)=\varphi_k(s,a)-\frac{1}{a_k}B_k$. Taking norms, each of the $\varepsilon_k$ terms can be bounded by inequality (10), and upper bound is independent on the explicit choice of $\varphi$, depending only on the regulator $R_\sigma$. With the choice of $\sigma$ given in the theorem, each intergral involving $\varepsilon_K$ converges to zero, while the other clearly converges to $I+B_k$. Therefore, $M(\varphi,a)\to M_0$ uniformly for $\varphi\in R_\sigma$. The result follows.

In practice, computing the time-scale tolerance is difficult. We can, thankfully, resort to conservative estimates.

**Theorem 4.** *Let $R$ be an exponential regulator for* (8)*. Suppose there exists a continuous function $n:\overline{S_c^*}\to\mathbb{R}_+$ such that $n(0)=0$ and*

$$||M(\varphi,a)-M_0||\leq n(a)$$

*for all $a\in S_c^*$ and $\varphi\in R$.*

1. *Let $\rho_\varepsilon M$ denote the $\varepsilon$-pseudospectral radius of the matrix $M$. The following inclusion is valid:*

$$\widehat{\mathscr{E}_s}(R):=\{a\in S_c^*:\rho_{n(a)}M_0<1\}\subseteq\mathscr{E}_s(R).$$

2. *Let $h>0$ denote the unique solution of the equation $\rho_h M_0=1$. The inequality*

$$\widehat{\mathscr{E}_t}(R):=\sup\{||a||:a\in B_{||a||}(0)\cap\widehat{\mathscr{E}_s}(R)\neq\emptyset\}\leq\mathscr{E}_t(R)$$

*is valid, and if $n$ is monotone increasing, we have $\widehat{\mathscr{E}_t}(R)=\min\{||a||:n(a)=h,\ a\in\overline{S_c^*}\}$.*

*Proof (Outline).* By definition of the pseudospectral radius, we have

$$\begin{aligned}\rho_{n(a)}M_0&=\sup\{\rho(Z):Z\in\mathbb{R}^{n\times n},||Z-M_0||\leq n(a)\}\\&\geq\sup\{\rho M(\varphi,a):\varphi\in R,\ ||M(\varphi,a)-M_0||\leq n(a)\}\\&=\sup\{\rho M(\varphi,a):\varphi\in R\}=\rho(R,a),\end{aligned}$$

which demonstrates the set inclusion. As for the inequality, that the supremum term is bounded by $\mathscr{E}_t(R)$ is obvious from the set inclusion. That $\widehat{\mathscr{E}_t}(R)$ is achieved at some $a$ for which $n(a)=h$ can be seen by noticing that, as $n$ is continuous and increasing, the set $\widehat{\mathscr{E}_s}(R)$ is star convex with basepoint 0. Consequently, maximizing the radius of a ball in the positive orthant within this set is equivalent to minimizing the distance to the boundary, and the latter is is precisely the level set $n(a)=h$.

**Corollary 2.** *Denote $X(t)=X(t;\tau_0)$. If $c=1$, the following inequality holds for all $\varphi\in R=(\sigma,w)$.*

$$||M(\varphi,a)-M_0|| \leq ||X(\tau_1)|| \left[\int_{S_0(a)} ||X^{-1}(s)|| \frac{w_0(s,a)}{e^{\sigma a_0}-1} ds + \left|\left| \frac{1}{a_0}\int_{S_0(a)} (X^{-1}(s)-I) ds B_0 \right|\right| \right]. \tag{14}$$

*Proof (Outline).*

$$M(\varphi,a)-M_0 = X(\tau_1)\left[I+\int_{S_0(a)} X^{-1}(s)\left(\varphi(s,a)-\frac{1}{a_0}B_0+\frac{1}{a_0}B_0\right)ds\right]-X(\tau_1)[I+B_0].$$

Re-arranging the above, taking norms and using inequality (10) provides the result.

If $c \neq 1$, a similar estimate to the above holds. However, it is rather cumbersome, and the associated proof is a notationally difficult inductive argument. It is omitted for brevity.

### 3.1 Example: An exact computation for a scalar equation

Consider the following scalar impulsive differential equation

$$\begin{aligned} x' &= \sigma x, \quad t \neq kT \\ \Delta x &= -bx, \quad t = kT, \end{aligned} \tag{15}$$

with parameters $\sigma > 0$, $b > 0$ and $T > 0$. Assume $M_0 = (1-b)e^{\sigma T} < 1$, so that the trivial solution is asymptotically stable. We choose $w(t,a) = c\left(\frac{a}{T}\right)^{\frac{1}{p}}$ for parameters $c$ and $p > 0$. The bound on the right-hand side of (14), denote $\tilde{n}(a)$, itself has an upper bound:

$$\tilde{n}(a) \leq n(a,p) := \frac{e^{\sigma T}}{\sigma} c \left(\frac{a}{T}\right)^{\frac{1}{p}} + e^{\sigma T} b \left(1 - \frac{1-e^{-\sigma a}}{\sigma a}\right),$$

and $n$ is strictly increasing in both $a$ and the parameter $p$. Therefore,

$$\rho_{n(a,p)} \leq \rho_{n(a,\infty)} = \lim_{p\to\infty} \rho_{n(a,p)} M_0 = e^{\sigma T}\left(1+\frac{c}{\sigma}-\frac{b}{\sigma a}(1-e^{-\sigma a})\right) \tag{16}$$

for each finite $p > 0$. Solving the equation $\rho_{n(a^*,\infty)}M_0 = 1$ for $a^*$ and applying Theorem 4, the following theorem is proven.

**Theorem 5.** *Consider the impulsive system* (15)*. Define* $u := \frac{1}{b}\left(e^{-\sigma T} - 1 - \frac{c}{\sigma}\right)$*. If* $M_0 := (1-b)e^{\sigma T} < 1$ *and* $c < (1-M_0)\sigma e^{-\sigma T}$*, then, for all* $a > 0$ *satisfying the inequality*

$$a < \frac{1}{\sigma}\left(W\left(\frac{1}{u}e^{\frac{1}{u}}\right)-\frac{1}{u}\right) := a^*,$$

*we have* $\rho M(\varphi,a) < 1$*, for all* $\varphi \in (\sigma,w)$*, with* $w(t,a) = c\left(\frac{a}{T}\right)^{\frac{1}{p}}$*, for any* $p > 0$*, where* $W$ *is the principal branch of the Lambert W function, or product logarithm function (ie. the inverse of the map* $x \mapsto xe^x$*).*

### 3.2 Example: Control of a pest with age structure

Consider the following system of impulsive differential equations.

$$X' = \begin{bmatrix} -10/21 & 5/7 \\ 1/4 & -1/7 \end{bmatrix} X, \qquad t \neq \tau_k \tag{17}$$

$$\Delta X = \begin{bmatrix} -0.7 & 0 \\ 0 & -0.4 \end{bmatrix} X, \qquad t = \tau_{2k} \tag{18}$$

$$\Delta X = \begin{bmatrix} 0 & 0 \\ 0 & -0.7 \end{bmatrix} X, \qquad t = \tau_{2k+1}, \tag{19}$$

with sequence of impulses $\tau_k = 7\lfloor k/2 \rfloor + (k \mod 2)$ and $t$ in units of days. The continuous dynamics, (17), could describe, for example, the population of some pest organism, $X = (X_1, X_2) \geq 0$, with juvenile ($X_1$) and adult ($X_2$) life stages. With both impulsive controls included, (17)–(19) is asymptotically stable, with dominant Floquet multiplier equal to 0.4200. If one or both controls are neglected, however, the trivial solution is unstable, so both controls are necessary to control the population.

Figure 1 provides visualizations of the subsets $\widehat{\mathscr{E}_s} \subseteq \mathscr{E}_s$ described in Theorem 4, of the $R$-stable sets for two uniform exponential regulators for the system (17)–(19). The first regulator, which generates the smaller of the two stable sets (red in the figure), is $R = (\sigma, \sqrt{a_k})$. The second regulator is $R = (\sigma, a_k)$. For both regulators, $\sigma = ||\Lambda||$, as in Corollary 1.

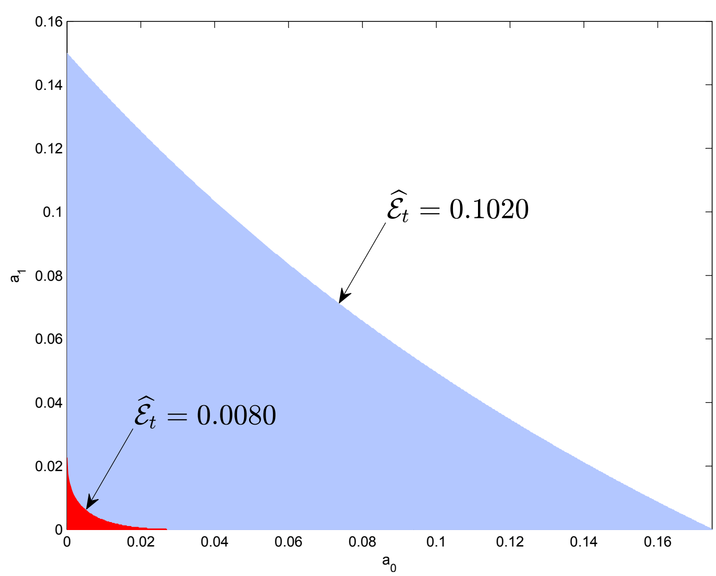


**Fig. 1** Conservative approximations, $\widehat{\mathscr{E}_s}$, of the $R$-stable sets for two uniform exponential regulators. Arrows indicate the associated lower bounds for time-scale tolerances.

### 3.3 Limitations

There are two main limitations of the techniques described in these proceedings. First and foremost, only linear systems are treated. The time-scale tolerance can indeed be defined for nonlinear systems of impulsive differential equations in more abstract settings, although the definitions must all be localized around periodic orbits or other stable objects. Some of our current ressearch concerns these problems.

Second, we treated only impulsive systems that are asymptotically stable. These techniques generally fail in the presence of a center subspace; see Example 3.5.6 of [2]. However, a similar approach does work if there is an unstable subspace. For example, if $\rho M_0 > 1$, one might want to know conditions on $a \in S_c^*$ under which $\rho M(\varphi,a) > 1$, for all $\varphi \in R$, with $R$ some suitable set of impulse extensions. If, for some continuous function $n$ satisfying $n(0) = 0$, we have $||M(\varphi,a) - M_0|| \leq n(a)$ for all $\varphi \in R$, one can verify the string of inequalities

$$\begin{aligned}\inf_{\varphi\in R}\rho M(\varphi,a) &\geq \inf\left\{\rho M : ||M - M_0|| \leq \sup_{\varphi\in R}||M(\varphi,a) - M_0||\right\}\\ &= \inf\{\rho M : ||M - M_0|| \leq \inf\{x : ||M(\varphi,a) - M_0|| \leq x,\ \forall\varphi \in R\}\}\\ &\geq \inf\{\rho M : ||M - M_0|| \leq n(a)\} := \rho^-_{n(a)} M_0\end{aligned}$$

holds. Therefore, an appropriate lower estimate for the analoguous time-scale tolerance can be found by solving the optimization problem

$$\widehat{\mathscr{E}}_t(R) := \sup\{||a|| : a \in B_{||a||}(0) \cap \{a \in S_c^* : \rho^-_{n(a)} M_0 > 1\} \neq \emptyset\}.$$

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