Linearization and local topological conjugacies for impulsive systems

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1 Introduction and background

Impulsive differential equations see applications in numerous fields where the systems of study exhibit rapid jumps in state. Such jumps may be intrinsic to the system, such as in the firing of a neuron in a biological neural network, or synthetic, such as the application of an insecticide or antibiotic treatment in a biological model. Arguably, one of the most common applications of the theory of impulsive differential equations arises in the latter case, where a continuous autonomous system is perturbed by impulses in an impulsive control setting. Specifically, there are many applications involving systems of the form

$$\dot{x} = Ax + f(x), \qquad t \neq \tau_k \tag{1}$$

$$\Delta x = Bx + g(x), \qquad t = \tau_k, \tag{2}$$

where (1) describes the continuous evolution of the system, and (2) the discontinuous impulsive control. We will call such a system an *impulsive system with autonomous right-hand side*. Frequently in applications, the sequence τ_k is periodic, and in this case, there are a wealth of techniques available to study the complete impulsive system. Specifically, the following elementary tools are available.

1. A linearized stability theorem.
2. A local bifurcation theory of fixed points and periodic orbits.
3. A local topological equivalence (Hartman-Grobman) theorem.

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Linearized stability principles for finite-dimensional impulsive systems have been known for quite some time; see the monograph [1] and the literature cited theirein for relevant background. In applications, Poincaré maps have been used to study bifurcations of fixed points and periodic orbits. There are also several generalizations of the Hartman-Grobman theorem to impulsive systems [3, 4, 5]. However, the latter results and seemingly all published Hartman-Grobman-type linearization theorems for impulsive systems assume global boundedness and Lipschitz conditions on the vector field and jump map, as well as a detailed analysis of the linearization. In these proceedings, we show how these assumptions can be weakened in favour of local smoothness assumptions, resulting in a direct analogue of the local, clasical Hartman-Grobman theorem.

1.1 The linearization theorem of Fenner and Pinto

The starting point for our result will be the linearization theorem of Fenner and Pinto [4]. We will later show how this theorem can be localized near an equilibrium point. To begin, we remind the reader of the definition of exponential dichotomy.

Definition 1. Consider the linear impulsive system

$$\begin{aligned} \dot{x} &= A(t)x, \quad t \neq \tau_k \\ \Delta x &= B_k x, \qquad t = \tau_k, \end{aligned} \tag{3}$$

with Cauchy (fundamental) matrix solution $X(t,s)$. We say the system (3) has *exponential dichotomy* if there exist $K \geq 1$, $\alpha > 0$ and a family $P(t)$ of projection matrices such that for all $t \geq s$ one has $X(t,s)P(s) = P(t)X(t,s)$ and the following inequalities.

$$\begin{aligned} ||X(t,s)P(s)|| &\leq Ke^{-\alpha(t-s)} \\ ||X(t,s)^{-1}[I-P(t)]|| &\leq Ke^{-\alpha(t-s)} \end{aligned}$$

The K and α are admissible *constants* and *exponents* of the dichotomy.

We briefly recall that in the event that (3) is periodic, the existence of an exponential dichotomy is equivalent to an eigenvalue condition. Namely, if one computes the monodromy matrix $M = X(\tau_0 + T, \tau_0)$ for period $T > 0$, the linear system has exponential dichotomy if and only if M has no eigenvalues with unit modulus [1]. Exponential dichotomy is also guaranteed if (3) is exponentialy stable, regardless of periodicity assumptions.

We will also need a notion of accumulation of the sequence of impulses. We will say that $\{\tau_k\}$ has *upper density bound* N if $\#\{\tau_k \in [n, n+1) : n \in \mathbb{Z}\} \leq N$. That is, in each unit interval starting at an integer, there are at most N impulse times.

Fenner and Pinto's linearization theorem [4] is stated in terms of (h,k)-dichotomies, which are more general than exponential dichotomies. To keep the presentation elementary, we will refrain from using this notion here. Then, a special case of the aforementioned theorem is as follows: it is stated with respect to the quasilinear system

$$\dot{x} = A(t)x + f(t,x), \quad t \neq \tau_k \tag{4}$$
$$\Delta x = B_k x + g_k(x), \qquad t = \tau_k, \tag{5}$$

and its *formal linearization*

$$\dot{x} = A(t)x, \quad t \neq \tau_k \tag{6}$$
$$\Delta x = B_k x, \qquad t = \tau_k. \tag{7}$$

Note that, here, it is not necessary for the functions f and g_k to satisfy $f(t,0) = g_k(0) = 0$; that is, 0 need not be an equilibrium of the nonlinear system (4)–(5). Additionally, f and g_k need not be nonlinear.

Proposition 1 (Fenner and Pinto, [4]). *Consider the periodic impulsive system* (4)–(5) *and the linear equation* (6)–(7). *Suppose the linear system has exponential dichotomy with constant K and exponent α. Assume for all $x,y \in \mathbb{R}^n$, the following inequalities are satisfied:*

$$||f(t,x)||| \leq \mu, \quad ||f(t,x) - f(t,y)|| \leq \gamma||x-y||,$$
$$||g_k(x)|| \leq \mu, \qquad ||g_k(x) - g_k(y)|| \leq \gamma||x-y||.$$

If the sequence $\{\tau_k\}$ of impulse times has upper density bound N and the inequality

$$2K\gamma\left(\frac{1}{\alpha} + N\frac{2-e^{-\alpha}}{1-e^{-\alpha}}\right) < 1 \tag{8}$$

holds, then there exists a function $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:

1. *For all $t \in \mathbb{R}$, $x \mapsto H_t(x) = H(t,x)$ is a homeomorphism,*
2. *$H_t(x(t))$ is a solution of the linear equation whenever $x(t)$ is a solution of the quasilinear equation,*
3. *$H_t^{-1}(y(t))$ is a solution of the quasilinear equation whenever $y(t)$ is a solution of the linear equation,*
4. *For all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, $H_t(x) - x$ is uniformly bounded with*

$$|H_t(x) - x| \leq 4K\mu\left(\frac{1}{\alpha} + N\frac{2-e^{-\alpha}}{1-e^{-\alpha}}\right). \tag{9}$$

Clearly, if the conditions of the proposition are satisfied, then the quasilinear system (4)–(5) admits at most one solution that is bounded for all time. This is to be expected, since the time-varying topological conjugacy H_t induces a *global* equivalence of the quasilinear system with its formal linearization, and the latter is exponentially dichotomous and therefore admits only one bounded solution: the trivial solution.

2 Localized impulsive linearization

In this section, we demonstrate how Proposition 1 can be localized near a given fixed point. The result is that one needs no longer verify the inequality (8), and so the constant and exponent of the exponential dichotomy need not be known explicitly. Next, we obtain a parameter-dependent analogue.

2.1 Main result and proof

Given the quasilinear system (4)–(5), we will say that the formal linearization (6)–(7) is *hyperbolic* if it has exponential dichotomy.

Theorem 1. *Let $f:\mathbb{R}\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ be differentiable in its second variable and let $g_k:\mathbb{R}^n\rightarrow\mathbb{R}^n$ be differentiable. Assume $t\mapsto Df(t,r(t))$ is locally integrable for all continuous functions $r(t)$ sufficiently small, and for each $\alpha\in\mathbb{R}$ there exists a continuously differentiable $H_\alpha:U\rightarrow\mathbb{R}^+$ for some open set U containing the origin, satisfying $H_\alpha(0)=0$ and $DH_\alpha(0)=0$, and a continuous $N_\alpha:U\rightarrow\mathbb{R}^+$ with $N_\alpha(0)=0$ such that*

$$\sup_{t\in[\alpha,\infty)}|f(t,x)|+\sup_{\tau_k\geq\alpha}|g_k(x)|\leq H_\alpha(x), \tag{10}$$

$$\sup_{t\in[\alpha,\infty)}||Df(t,x)|+\sup_{\tau_k\geq\alpha}||Dg_k(x)||\leq N_\alpha(x) \tag{11}$$

on U. Consider the impulsive system (4)–(5) *with process φ and the linearized equation* (6)–(7) *with process L. Suppose the linearization is hyperbolic and the sequence of impulses has an upper density bound. Let $\xi>0$ be given. There exists a family of homeomorphisms $H_t:\mathbb{R}^n\rightarrow\mathbb{R}^n$, $t\in\mathbb{R}$, satisfying the bound $|H_tx-x|\leq\xi$ and the following additional properties:*

- *For all $u\leq v\in\mathbb{R}$, there exists $\delta>0$ such that for all $||x||<\delta$, one has $H_t\circ\varphi(t,s)x=L(t,s)H_sx$ for all $[s,t]\subseteq[u,v]$.*
- *If $x^*=0$ is either linearly or nonlinearly stable, then for all $s\in\mathbb{R}$, there exists $\delta>0$ such that $H_t\circ\varphi(t,s)x=L(t,s)H_sx$ for $||x||<\delta$ and $s\leq t<\infty$.*

Proof. Let $\varepsilon>0$ be a constant that has yet to be chosen. By the mean-value theorem and the assumptions on f and g_k, one can show that

$$||\varphi(t,u,x)||\leq||x||+\int_u^t M_{u,v}||\varphi(r,u,x)||dr+\sum_{u\leq\tau_k<t}M_{u,v}||\varphi(\tau_k,u,x)||,$$
$$M_{u,v}=\sup_{[u,v]}||A(t)||+\max_{\tau_k\in[u,v]}||B_k||+\max_{||x||\leq\varepsilon}F_u(x),$$

provided $||x||\leq\varepsilon_2<\varepsilon$ for some ε_2 and $t>u$ is small enough. By Gronwall's inequality for impulsive systems [1], it follows that

$$||\varphi(t,s,x)|| \leq ||x||(1+M_{u,v})^{N(\lceil t\rceil-\lfloor s\rfloor)}e^{M_{u,v}(t-s)} := ||x||C_{s,t}$$

for $t-s>0$ small enough. One obtains $||\varphi(t,s,x)|| \leq \varepsilon$ for $[s,t]\subseteq[u,v]$ provided $||x|| \leq \varepsilon_2 = \varepsilon C_{u,v}^{-1}$.

Let $r^\varepsilon : U \to [0,1]$ be a smooth cutoff function satisfying $r^\varepsilon|_{||x||\leq\varepsilon} = \text{Id}$, $r^\varepsilon|_{||x||\geq 2\varepsilon} = 0$ and $||\nabla r^\varepsilon|| \leq 2/\varepsilon$ on the annulus $\varepsilon \leq ||x|| \leq 2\varepsilon$, where we assume without loss of generality that U contains the closed ball of radius 2ε centered at the origin. Define the time-varying cutoff vector fields and jump maps as follows:

$$f^\varepsilon(t,x) = \begin{cases} f(t,x)r^\varepsilon(x), & t\geq s \\ 0 & t<s \end{cases} \qquad g_k^\varepsilon(x) = \begin{cases} g_k(x)r^\varepsilon(x), & s\leq\tau_k \\ 0 & s>\tau_k \end{cases}$$

By the mean-value theorem and the chain rule, we obtain the estimates

$$\max\{||Df^\varepsilon||,||Dg_k^\varepsilon||\} \leq \sup_{||x||\leq 2\varepsilon}||N_u(x)|| + 4\sup_{||x||\leq 2\varepsilon}||DH_u|| := A(\varepsilon)$$

uniformly, for all $t\in\mathbb{R}$, $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$. Choose ε small enough so that inequality (8) is satisfied with $\gamma = A(\varepsilon)$ and apply Proposition 1 to the quasilinear system

$$\begin{aligned} \dot{x} &= A(t)x + f^\varepsilon(t,x), && t\neq\tau_k \\ \Delta x &= B_k x + g_k^\varepsilon(x), && t=\tau_k \end{aligned}$$

to obtain the family of conjugacies H_t. Then, $H_t\circ\varphi^\varepsilon(t,s)x = L(t,s)\circ H_s x$ for all $x\in\mathbb{R}$ and $t\geq s$. In particular, it holds for $||x|| < \delta = \varepsilon C_{u,v}^{-1}$ and $[s,t]\subseteq[u,v]$. Since it is known that $||\varphi(t,s)x|| \leq \varepsilon$ whenever these same constraints hold, we obtain the equality $H_t\circ\varphi(t,s)x = L(t,s)\circ H_s x$ for $||x||\delta$ and $[s,t]\subseteq[u,v]$, as claimed in the theorem.

Now we prove the assertions concerning stability. We prove only that linear stability implies nonlinear stability, since the other direction is similar. Suppose $x^*=0$ is linearly stable. To begin, let some $s\in\mathbb{R}$ be given and let $\varepsilon>0$, where we may assume without loss of generality that ε is small enough for inequality (8) holds with $\gamma = A(\varepsilon)$. By linear stability, let $\delta_1>0$ be small enough so that $\rho = \sup_{||y||\leq\delta_1}||L(t,s)y||_{t\geq s}$ satisfies inequality

$$4K\rho\left(\frac{1}{\alpha}+N\frac{2-e^{-\alpha}}{1-e^{-\alpha}}\right) < \frac{\varepsilon}{2}.$$

By triangle inequality and the conjugation property, we have

$$||\varphi_\varepsilon(t,s)x|| \leq ||(H_t^{-1}-id)L(t,s)H_s x|| + ||L(t,s)H_s x||. \tag{12}$$

From inequality (9), it follows that for $||y||\leq\delta_1$ one has $||H_t y - y|| \leq \varepsilon/2$, and by homeomorphism, one obtains $||H_t^{-1}y-y|| \leq \varepsilon/2$ as well. By continuity of H_s, the same holds true with $y = H_s x$ provided $||x||$ is small enough. Stability of $L(t,s)$ and continuity of H_s once again yield $||L(t,s)H_s x|| < \varepsilon/2$ for $||x||$ small enough. It then follows that there exists some $\delta>0$ such that, in light of inequality (12),

$||\varphi_\varepsilon(t,s)x|| < \varepsilon$ whenever $||x|| < \delta$ and $t \geq s$. But this implies that $\varphi_\varepsilon(t,s)x = \varphi(t,s)x$ for all such $||x|| < \delta$.

In the periodic case, the inequality conditions of Theorem 1 are satisfied under reasonable smoothness assumptions on the vector field and jump map. The following corollary makes this concrete, and the proof is obvious.

Corollary 1. *Inequalities* (10)–(11) *can be replaced with one of the following stronger assumptions.*

- *The quasilinear system* (4)–(5) *is periodic, each* g_k *is* C^1*, and* f *and* Df *are continuous at* (t,x) *unless* $t = \tau_k$*, where they are continuous from the right and have left limits in* t.
- *The "nonlinearities"* f *and* g_k *of the quasilinear system are autonomous and* C^1.

2.2 An application to bifurcation theory

Consider the system

$$\dot{x} = f(t,x,\lambda), \quad t \neq \tau_k \tag{13}$$
$$\Delta x = g_k(x,\lambda), \quad t = \tau_k, \tag{14}$$

together with its linearization at the origin

$$\dot{y} = Df(0,t,\lambda)y, \quad t \neq \tau_k \tag{15}$$
$$\Delta y = Dg_k(0,\lambda)y, \quad t = \tau_k. \tag{16}$$

dependent on a parameter $p \in \Pi \subseteq \mathbb{R}^m$, and suppose that 0 is an equilibrium point for all $p \in B_r(\pi)$ for some $r > 0$ and some given $\pi \in \Pi$. Generally, if one posesses a parameter-dependent family of periodic orbits $x_p^*(t)$ or equilibrium points, the time- and parameter-dependent change of variables $x = y + x_p^*$ transforms the system to one in which 0 is an equilibrium point, independent of p. One then has the following theorem.

Theorem 2. *Suppose* $f : \mathbb{R} \times \mathbb{R}^n \times \Pi \to \mathbb{R}^n$ *and* $g_k : \mathbb{R}^n \times \Pi \to \mathbb{R}^n$ *are* C^1 *and* $f(t,0,p) = g_k(0,p) = 0$ *for all* $t \in \mathbb{R}$ *and* $k \in \mathbb{Z}$*; that is,* 0 *is an equilibrium point of* (13)–(14)*. Let* $\pi \in \Pi$ *be given, and assue for each* $\alpha \in \mathbb{R}$*, there exists* $P \subseteq \Pi$ *with* $\pi \in P$*, a continuously differentiable* $H_\alpha : U \times P \to \mathbb{R}^+$ *satisfying* $H_\alpha(0,p) = 0$ *and* $DH_\alpha(0,p) = 0$*, and a continuous* $N_\alpha : U \times P \to \mathbb{R}^+$ *satisfying* $N_\alpha(0,p) = 0$*, such that*

$$\sup_{t\in[\alpha,\infty)} |F(t,x,p)| + \sup_{\tau_k \geq \alpha} |G_k(x,p)| \leq H_\alpha(x,p), \tag{17}$$

$$\sup_{t\in[\alpha,\infty)} ||DF(t,x,p)| + \sup_{\tau_k \geq \alpha} ||DG_k(x,p)|| \leq N_\alpha(x,p) \tag{18}$$

on $\mathbb{R}^n \times P$, where $F(t,x,p) = f(t,x,p) - Df(t,0,\pi)x$ and $G_k(x,p) = g_k(x,p) - Dg_k(0,\pi)x$. For a given parameter $p \in \Pi$, let φ^p denote the process associated to the impulsive system (13)–(14)*, and L^p the process of its associated linearized equation* (15)–(16)*. Suppose L^π is hyperbolic and the sequence τ_k has a upper density bound. Then, there exists $\eta > 0$ such that for all $p_1, p_2 \in (\pi - \eta, \pi + \eta)$, there exists a family of homeomorphisms $H_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{R}$, with the following properties:*

- *For all $[u,v] \subset \mathbb{R}$ bounded, there exists $\delta > 0$ such that for all $||x|| < \delta$, one has $H_t \circ \varphi^{p_1}(t,s)x = \varphi^{p_2}(t,s)H_s x$ for all $[s,t] \subseteq [u,v]$.*
- *If $x^* = 0$ is linearly stable at parameter π, the previous conclusion holds, but the interval $[u,v]$ can be unbounded on the right.*

Proof. Write system (4)–(5) equivalently, for each paramter p, let $E(p)$ denote the system,

$$\begin{aligned} \dot{x} &= Df(t,0,\pi)x + F(t,x,p), \quad t \neq \tau_k \\ \Delta x &= Dg_k(0,\pi)x + G_k(x,p), \qquad t = \tau_k, \end{aligned}$$

where $F(t,x,p) = f(t,x,p) - Df(t,0,\pi)x$ and $G_k(x,p) = g_k(x,p) - Dg_k(0,\pi)x$. Also, denote by $L(\pi)$ its formal linearization

$$\begin{aligned} \dot{x} &= Df(t,0,\pi)x, \quad t \neq \tau_k \\ \Delta x &= Dg_k(0,\pi)x, \qquad t = \tau_k, \end{aligned}$$

Broadly, the idea of the proof is as follows. We locally conjugate the process associated to $E(p_1)$ to $L(\pi)$ by a family of homeomorphism H_t, and also conjugate the process associated to $E(p_2)$ to $L(\pi)$ by a family of homeomorphism G_t. The family of compositions $G_t^{-1} \circ H_t$ will then be a local conjugacy of $E(p_1)$ with $E(p_2)$. Therefore, it is enough to emulate the proof of Theorem 1 with a parameter, establishing the existence of the conjugacy of $E(p_1)$ with $L(\pi)$ provided $|p_1 - \pi|$ is small enough, since the other conjugacy is obtained by the same method.

Let φ denote the process associated with $E(p_1)$. Let $\varepsilon > 0$ be a constant that has yet to be chosen. One has

$$\begin{aligned} ||\varphi(t,u,x)|| &\leq ||x|| + \int_u^t M(p)||\varphi(r,u,x)||dr + \sum_{u \leq \tau_k < t} M(p)||\varphi(\tau_k,u,x)||, \\ M(p) &= \sup_{t \in [u,v]} ||Df(t,0,\pi)|| + \max_{\tau_k \in [u,v]} ||Dg_k(0,\pi)|| \\ &\quad + \sup_{||x|| \leq \varepsilon} H_u(x,p) + \sup_{||x|| \leq \varepsilon} N_u(x,p), \end{aligned}$$

provided $||x|| \leq \varepsilon_2 < \varepsilon$ for some ε_2 and $t > u$ is small enough. By Gronwall's inequality for impulsive systems [1], it follows that

$$||\varphi(t,s,x)|| \leq ||x||(1 + M(p))^{N(\lceil t \rceil - \lfloor s \rfloor)} e^{M(p)(t-s)} := ||x||C_{s,t}(p)$$

for $t > s$ small enough. One obtains $||\varphi(t,s,x)|| \leq \varepsilon$ for $[s,t] \subseteq [u,v]$ provided $||x|| \leq \varepsilon_2 = \varepsilon C_{u,v}(p)^{-1}$. As in a previous proof, set

$$A(\varepsilon,\eta) = \sup_{(x,p)\in D} ||N_u(x,p)|| + 4 \sup_{(x,p)\in D} ||DH_u(x,p)||$$

with $D = \{(x,p) : ||x|| \leq 2\varepsilon, |p-\pi| \leq \eta\}$, and choose $\eta > 0$ and $\varepsilon > 0$ small enough so that inequality (8) is satisfied with $\gamma = A(\varepsilon,\eta)$ for all $||x|| \leq 2\varepsilon$ and $|p-\pi| < \eta$. Defining the cutoff process φ_ε at this $\varepsilon > 0$ and choosing $p_1 \in (\pi-\eta, \pi+\eta)$, the proof proceeds in essentially the same way as that of Theorem 1, with the only difference being that η and ε may need to be chosen smaller to accomodate for the restrictions $|F(t,x,p_1)| \leq \mu$ and $|G_k(x,p_1)| \leq \mu$ for the second point concerning stability. Again, the argument is analogous to the one for Theorem 1, and is omitted.

An obvious variant of Corollary 1 holds for Theorem 2. Namely, we have the following.

Corollary 2. *Inequalities* (17)–(18) *can be replaced with one of the following stronger assumptions.*

- *The nonlinear system* (13)–(14) *is periodic, each $g_k(x,p)$ is C^1, f and Df are continuous at (t,x,p) unless $t = \tau_k$, where they are continuous from the right and have left limits in t.*
- *The vector field and jump map f and g_k of the quasilinear system are autonomous and C^1*

3 Discussion

Theorem 1 provides a local topological conjugacy theorem for hyperbolic fixed points of finite-dimensional impulsive differential equations. The result is also applicable to hyperbolic periodic orbits and bounded solutions by first applying the changes of coordinates to map these orbits to the origin. The result generalizes the more classical Hartman-Grobman theorem, which assumes only local properties such as smoothness of the vector field, rather than global Lipschitz conditions.

Theorem 2 is a robustness result. Qualitatively, the consequence of the theorem is that hyperbolic periodic orbits and equilibrium points do not bifurcate as their parameters are varied; their phase portraits are locally conjugate for all sufficiently nearby parameters. Technically, we have not proven this for periodic orbits explicitly, since one needs to know a priori that the family of periodic orbits x_p^* varies continuously with respect to the parameter p. However, this is always the case provided the periodic orbit is hyperbolic at the reference parameter π; one may consult Proposition 3 of Church and Liu [2] for additional details.

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