Invariant manifold-guided impulsive stabilization of delay equations

Kevin E.M. Church, Xinzhi Liu

***Abstract*—We propose an impulsive stabilization method for delay equations based on a reduction to the centre-unstable manifold. Our approach does not make use of the constructions of Lyapunov's direct method, such as Lyapunov functions or functionals. Necessary and sufficient conditions for when a system can be stabilized by impulses are derived based on these ideas, which subsequently lead to impulsive stabilization heuristics that are amenable to implementation in a scientific computing environment. Two examples are provided to illustrate the results.**

I. Introduction

Stability is one of the fundamental requirements in system design, analysis and control. In recent decades, impulsive control has been shown to be a powerful approach to stabilize dynamical systems, and various stability and stabilization results have been obtained. The Lyapunov method has been a long-lasting and powerful tool in the study of stability problems in dynamical systems, and it has been used extensively for stabilization in impulsive dynamical systems with time delays. This stabilization methodology is now known as impulsive stabilization and many interesting results have been obtained, see [1]–[8]. Most of these are proven using Lyapunov functionals (or functions, as in the Lyapunov-Razhumikhin method) and are stated in terms of existence of matrices satisfying linear matrix inequalities. Despite its advantages there are some known weakness and difficulties in the Lyapunov-based approach. First of all, it can be difficult to construct a Lyapunov function or Lyapunov functional that is able to prove stability of a fixed point in a particular system. Different choices of Lyapunov functions/functionals also give rise to different sufficient conditions for stabilization, and it can be difficult to ascertain the best choice for a given problem. Moreover, many Lyapunov-based results tend to be conservative. These difficulties motivate us to consider an alternative approach for impulsive stabilization.

From a theoretical standpoint, it remains unclear under what conditions it is possible to stabilize a delay differential equation by impulses. Since the former equations are infinite-dimensional and impulses act at isolated points in time, this is by no means a trivial problem. Conditions for stabilization by impulses have been developed by several authors – see for instance the list of references in the previous paragraph – but these are all derived by way of a particular Lyapunov function or functional and therefore only provide a way to test whether a particular impulsive controller guarantees stability. The lack of necessary conditions gives us further motivation to move away from stability and stabilization by way of Lyapunov's direct method.

The objective of this note is to provide an invariant manifold-guided impulsive stabilization approach for dynamical systems with time delays. We provide a formal problem statement in Section II. Section III contains some necessary background on linear impulsive delay systems. Section IV contains our abstract and concrete sufficient conditions for impulsive stabilization. Section V specializes the results to periodic impulses. A converse result concerning the feasibility of impulsive stabilization is formalized and proven in Section VI. Two examples are provided to illustrate the results: a three-dimensional system (Section VII) and a ten-dimensional coupled oscillator network (Section VIII). The paper ends with a conclusion, Section IX.

Notation: Given an interval I, the symbol $\mathcal{RCR}(I,X)$ for X a normed space will denote the set of functions $\phi : I \to X$ that are continuous from the right and possess limits on the left (i.e. $\lim_{t\to s^-}\phi(t)$ exists). When I is compact, the norm on this space will always be the supremum norm: $||\phi|| = \sup_{t\in I}|\phi(t)|$, where $|\cdot|$ is the norm on X. The left limit of a function ϕ at some time t will be denoted $\phi(t^-)$. If M is a square matrix, $\rho(M)$ denotes its spectral radius. If M is a real matrix then $M^\intercal$ is its transpose. $\mathbb{R}^{n*}$ will denote the canonical dual space of row vectors. For a function $F : X \to Y$, its image will be denoted $\mathrm{Im}(F)$. The range of a linear operator $L : X \to X$ will be denoted $\mathcal{R}(L)$. For $y \in \mathcal{RCR}(I,X)$, we denote $\Delta y(t) = y(t) - y(t^-)$. Let $\tau > 0$ be fixed; if $[t-\tau,t] \subset I$, we define $y_t(\theta) = y(t+\theta)$ for $\theta \in [-\tau,0]$, and $y_{t^-}(\theta) = y(t+\theta)$ for $\theta \in [-\tau,0)$, and $y_{t^-}(0) = y(t^-)$, so that $y_t \in \mathcal{RCR}([t,\sup I],X)$ and $y_{t^-} \in \mathcal{RCR}([t,\sup I],X)$. We write $\epsilon = O(g)$ for nonnegative $\epsilon = \epsilon(x)$ if there exists $C > 0$ such that $\epsilon(x) \le Cg$ for all $x \ge 0$ sufficiently small.

II. Problem statement

The starting point is a linear n-dimensional delay differential equation

$$\dot{y} = A_0 y(t) + \sum_{j=1}^{m} A_j y(t-\tau_j) \tag{2.1}$$

with delays $0 < \tau_1 < \cdots < \tau_m$ and matrices $L_j \in \mathbb{R}^{n\times n}$, for which the trivial equilibrium $0 \in \mathbb{R}^n$ that is presumably unstable. The goal is to find $\ell + 1$ sequences of matrices $\{B_{0,k},\dots,B_{\ell,k}\}_{k\in\mathbb{N}}$, a sequence of *impulse times* $\{t_k : k \in$

Kevin E.M. Church is at the Department of Mathematics and Statistics at McGill University, email: kevin.church@mcgill.ca. Xinzhi Liu is at the Department of Applied Mathematics at the University of Waterloo, email: xzliu@uwaterloo.ca.

$\mathbb{N}\}$ and delays $r_1, \ldots, r_\ell$ such that 0 is also an equilibrium in the impulsive system

$$\dot{y} = A_0 y(t) + \sum_{j=1}^{m} A_j y(t-\tau_j), \qquad t \neq t_k \tag{2.2}$$

$$\Delta y = B_{0,k} y(t^-) + \sum_{j=1}^{\ell} B_{j,k} y(t-r_j), \qquad t = t_k, \tag{2.3}$$

but the impulses cause it to become locally asymptotically stable, or at least causes its solutions to not grow as quickly. We later quantify this by requiring the impulses to decrease the *continuous growth rate*. Note that the delays $r_1 < \cdots < r_\ell$ in the impulsive part could be distinct from the $\tau_1, \ldots, \tau_m$.

Remark 1. An analogous problem can be formulated for the nonlinear version of (2.1), and the methods we propose here remain applicable: see [9] for details. Also, if the plant (2.1) has no delays and there are no impulsive delays (i.e. $\ell = 0$), the dynamical system generated by (2.2)–(2.3) becomes truly finite-dimensional and our method is not nearly as powerful, since the dimension reduction is from finite to finite.

III. Background: Linear systems

From this point forward, we set $\tau = \max\{\tau_m, r_\ell\}$ (the largest delay) and let $\mathcal{RCR} = \mathcal{RCR}([-\tau, 0], \mathbb{R}^n)$.

Definition 2. Let $s \in \mathbb{R}$. A function $x \in \mathcal{RCR}([s-\tau, \infty), \mathbb{R}^n)$ is a *solution* of (2.2)–(2.3) if it is differentiable from the right everywhere in $[s, \infty)$, its derivative satisfies (2.2), and at times $t \in \{t_k : k \in \mathbb{N}\} \cap (s, \infty)$ it satisfies (2.3).

Definition 3. A real number r is a *continuous growth rate for* (4.5)–(4.6) if there exists $K > 0$ such that every one of its solutions $x : [s-\tau, \infty) \to \mathbb{R}^n$ satisfies $|x(t)| \leq K e^{r(t-s)} ||x_s||$ for $t \geq s$. This growth rate is *tight* if any other growth rate r' satisfies $r' \geq r$.

Proposition 4. *The* impulse-free centre-unstable subspace*, denoted N, exists, is finite-dimensional, and is the set of all functions $\phi \in \mathcal{RCR}([-\tau, 0], \mathbb{R}^n)$ such that any solution $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ of the delay differential equation* (2.2) *(i.e. without impulses) with $x_0 = \phi$ uniquely decomposes into a sum $x = x_c + x_u$, where*

- *$|x_c(t)|$ grows at most polynomially as $|t| \to \infty$;*
- *$|x_u(t)| \leq K e^{ct}$ for all $t \in \mathbb{R}$, for some positive K, c.*

Remark 5. Backward continuation is not invoked or needed because the centre-unstable subspace is spanned by eigenfunctions on which the flow is global (and equivalent to an ordinary differential equation). See Section IV, Theorem 2.15 of [10].

Definition 6. The *(formally) adjoint impulse-free centre-unstable eigenspace*, denoted $N^\intercal$, exists, and is the set of all functions $\phi : [0, \tau] \to \mathbb{R}^{n*}$ such that any solution $w : \mathbb{R} \to \mathbb{R}^{n*}$ of

$$\dot{w} = -w(t) A_0^\intercal - \sum_{j=1}^{m} w(t+\tau_j) A_j^\intercal \tag{3.4}$$

with $w|_{[0,\tau]} = \phi$ uniquely decomposes as $w = w_c + w_u$, where

- $|w_c(t)|$ grows at most polynomially as $|t| \to \infty$;
- $|w_u(t)| \leq K e^{-ct}$ for all $t \in \mathbb{R}$, for some positive K, c.

Remark 7. In practice, it is not necessary to work with system (3.4) directly – see the adjoint theory from [11].

IV. General Sufficient conditions

Let $\ell \geq 0$ be fixed. Introduce the set

$$\mathbf{B} = \left\{ \{B_{0,k}, \ldots, B_{\ell,k}\}_{k \in \mathbb{N}} : \max_{j=0,\ldots,\ell} \sup_{k \geq 0} ||B_{j,k}|| < \infty \right\}.$$

It is a Banach space when equipped with the norm

$$||B||_{\mathbf{B}} = \max_{j=0,\ldots,\ell} \sup_{k \geq 0} ||B_{j,k}||.$$

The following result is at the core of our stabilization idea. It states that, heuristically, stabilization can be accomplished by designing the matrix sequence $\{B_{0,k}, \ldots, B_{\ell,k}\}_{k \in \mathbb{N}} \in \mathbf{B}$ to control the dynamics on the centre-unstable manifold. We will make this idea more explicit with Corollary 10.

Theorem 8. *Consider the impulsive delay differential equation*

$$\dot{x} = A_0 x(t) + \sum_{j=1}^{m} A_j x(t-\tau_j), \qquad t \neq t_k \tag{4.5}$$

$$\Delta x = B_k x_{t^-}, \qquad t = t_k, \tag{4.6}$$

where $B_k : \mathcal{RCR} \to \mathbb{R}^n$ is a functional of the form $B_k \zeta = B_{0,k}\zeta(0) + \sum_{j=1}^{\ell} B_{j,k}\zeta(-r_j)$ for $\{B_{0,m}, \ldots, B_{\ell,m}\}_{m \in \mathbb{N}} \in \mathbf{B}$. Suppose the impulse times and impulsive delays satisfy $t_k - r_j \neq t_m$ whenever $k > m \geq 0$ for all $j = 1, \ldots, \ell$. Suppose additionaly that $t_{k+1} - t_k \geq p$ for some $p > 0$. There exists $\delta > 0$ and a function $\mathcal{CU} : \mathbb{R} \times N \times \mathbf{B} \to \mathcal{RCR}$ with the following properties.

1) *The* parameter-dependent centre-unstable manifold
$$\mathcal{W}_{cu}(t; B) = \{\mathcal{CU}(t, \phi, B) : \phi \in N\}$$
has dimension $q = \dim(N)$. It is positively invariant under (4.5)–(4.6) *provided $||B||_{\mathbf{B}} < \delta$, and coincides precisely with the centre-unstable fibre bundle.*
2) *If $||B||_{\mathbf{B}} < \delta$ the dynamics of* (4.5)–(4.6) *restricted $\mathcal{W}_{cu}(t; B)$ are smoothly equivalent near $x = 0$ to the q-dimensional system without delays*
$$\dot{u} = \Lambda u, \qquad t \neq t_k \tag{4.7}$$
$$\Delta u = \Gamma \Psi(0) B_k \Phi_0 u(t^-), \qquad t = t_k, \tag{4.8}$$
where $\Phi(t)$ is a $n \times q$ matrix whose columns $\phi^{(i)}(t)$ for $i = 1, \ldots, q$ are real solutions of (4.5) *for which $\{\phi_0^{(1)}, \ldots, \phi_0^{(q)}\}$ is a basis for N, Λ is a real $q \times q$ matrix satisfying the equation $\frac{d\Phi(t)}{dt} = \Phi(t)\Lambda$, $\Psi(t)$ is a $q \times n$ matrix whose rows $\psi^{(i)}$ for $i = 1, \ldots, q$ are real solutions of* (3.4) *for which $\{\psi^{(1)}|_{[0,\tau]}, \ldots, \psi^{(q)}|_{[0,\tau]}\}$ is a basis for $N^\intercal$, and $\Gamma \in \mathbb{R}^{q \times q}$ is defined by*
$$\Gamma^{-1} = \Psi(0)\Phi(0) - \sum_{k=1}^{m} \int_0^{\tau_k} \Psi(s) A_k \Phi(s) ds. \tag{4.9}$$
3) *If $||B||_{\mathbf{B}} < \delta$ and* (4.7)–(4.8) *has continuous growth rate r, then $r + O(||B||_{\mathbf{B}}^2)$ is a continuous growth rate for* (4.5)–(4.6).

Proof. First, consider for $B \in \mathbf{B}$ fixed the following impulsive delay differential equation of dimension $n+1$:

$$\dot{x} = A_0 x(t) + \sum_{j=1}^{m} A_j x(t-\tau_j), \qquad t \neq t_k \tag{4.10}$$

$$\dot{v} = 0, \qquad t \neq t_k \tag{4.11}$$

$$\Delta x = v B_k x_{t^-}, \qquad t = t_k \tag{4.12}$$

$$\Delta v = 0, \qquad t = t_k, \tag{4.13}$$

where v is scalar. Here, $vB_k x_{t^-}$ is interpreted as a nonlinear term. The functional $g_k : \mathcal{RCR}([-\tau,0],\mathbb{R}^{n+1}) \to \mathbb{R}^n$ that defines this nonlinearity,

$$g_k(v,\phi) = v(0)B_k\phi,$$

is smooth. Also, for any fixed $\epsilon > 0$, one can check that g_k is Lipschitz continuous on the ball $B_\epsilon(0)$, with Lipschitz constant $2\epsilon||B||_{\mathbf{B}}$. Let $\xi : \mathbb{R} \to \mathbb{R}^+$ be a C^∞ cutoff function [12] satisfying $\xi(x) = 1$ for $|x| \leq 1$, $\xi(x) \leq 1$ for $1 < |x| < 2$, and $|x| = 0$ for $|x| \geq 2$. Cut off the nonlinearity $g_k(v,\phi)$ outside $B_{2\epsilon}(0)$ by replacing it with

$$\tilde{g}_k(v,\phi) = g_k(v,\phi)\xi\left(\frac{||P_{cu}(v,\phi)||}{\epsilon R}\right)\xi\left(\frac{||P_s(v,\phi)||}{\epsilon R}\right),$$

with $P_{cu} : \mathcal{RCR}([-\tau,0],\times\mathbb{R}^{n+1}) \to \mathcal{RCR}([-\tau,0],\times\mathbb{R}^{n+1})$ the projection onto the centre-unstable subspace of (4.10)–(4.13) with $B = 0$ considered as a delay differential equation, $P_s : \mathcal{RCR}([-\tau,0],\mathbb{R}^{n+1}) \to \mathcal{RCR}([-\tau,0],\mathbb{R}^{n+1})$ the projection onto the analogous stable subspace, and $R = \max\{||P_{cu}||, ||P_s||\}$. One can verify that there exists a constant $\kappa > 0$ that depends only on the choice of cutoff function ξ such that $\tilde{g}_k(v,\phi)$ is globally Lipschitz continuous with Lipschitz constant $2\epsilon\kappa||B||_{\mathbf{B}}$. Emulating the construction of the centre manifold from [13] and the unstable manifold from [14], one obtains the existence of a function $F_B : \mathbb{R} \times \mathcal{R}(P_{cu}) \to \mathcal{RCR}([-\tau,0],\mathbb{R}\times\mathbb{R}^n)$ for which the fibre bundle

$$\tilde{W}_{cu}(t;B) = \{F_B(t,(v,\phi)) : (v,\phi) \in \mathcal{R}(P_{cu})\}$$

is invariant under (4.10)–(4.13) for initial conditions that satisfy $||(v_0,\phi_0)|| < \delta'$ for some $\delta' > 0$. In particular, $\delta' > 0$ can be chosen by controlling only the size of the Lipschitz constant of the cutoff nonlinearity, ϵ. Consequently, if we fix some $\delta_0 > 0$, the above construction works for *any* $B \in \mathbf{B}$ that satisfies $||B||_{\mathbf{B}} \leq \delta_0$, and the aforementioned $\delta' > 0$ is independent of B. Consequently, if we define the projection $\pi : \mathcal{RCR}([-\tau,0],\mathbb{R}^{n+1}) \to \mathcal{RCR}$ by $\pi(v,\phi) = \phi$, then the fibre bundle

$$\mathcal{F}_{cu}(t;B) = \{\pi F_B(t,(v,\phi)) : (v,\phi) \in \mathcal{R}(P_{cu})\}$$

is locally positively (in x) invariant under the dynamics of (4.5)–(4.6) provided $||B||_{\mathbf{B}} < \delta_0$ and $|v| < \delta'$. However, since the equation is linear, the positive invariance is global.

Fix some $r \in (0,\delta_0)$ and consider $S_r(0) = \{B \in \mathbf{B} : ||B||_{\mathbf{B}} = r\}$. The ball $B_{r\delta'}(0) \subset \mathbf{B}$ is contained within

$$U := \{B \in \mathbf{B} : B = s\tilde{B},\ 0 \leq s < \delta', \tilde{B} \in S_r(0)\}$$

and $\mathcal{F}_{cu}(t;B)$ is invariant under the dynamics of (4.5)–(4.6) provided $B \in U$, in the sense that solutions $x \in \mathcal{RCR}([s-\tau,\infty),\mathbb{R}^n)$ with $||x_s|| < \delta'$ satisfy $x_t \in \mathcal{F}_{cu}(t;B)$ for $t \in [s,T]$ for some $T > 0$. Define $\delta = \min\{\delta', r\delta'\}$.

One can verify that $\mathcal{R}(P_{cu}) = Z \times N$, where $Z = \{v \in \mathcal{RCR}([-\tau,0],\mathbb{R}) : v \text{ is constant}\}$. Define the function $\mathcal{CU} : \mathbb{R} \times N \times \mathbf{B} \to \mathcal{RCR}$ by

$$\mathcal{CU}(t,\phi,B) = \pi F_{\widetilde{B}}(t,(r^{-1}||B||_{\mathbf{B}},\phi)), \quad \widetilde{B} = \frac{r}{||B||_{\mathbf{B}}}B,$$

unless $B = 0$, in which case we set $\mathcal{CU}(t,\phi,0) = \pi F_0(t,(0,\phi))$. If $0 < ||B||_{\mathbf{B}} < \delta$, then by previous observations B is uniquely associated with some $\tilde{B} \in S_r(0)$ and some $s \in [0,\delta')$. By inspection, $\tilde{B}$ is as defined above and $s = r^{-1}||B||_{\mathbf{B}}$. When $B = 0$ we recover the linear plant (2.1). It follows that the fibre bundle $\mathcal{W}_{cu}$ has the properties 1) of the theorem. Smoothness is a consequence of the smoothness of the centre-unstable manifold (see [14] for the centre and unstable cases separately; the unified case is similar).

The proof of 2) concerning the smoothly equivalent dynamics (4.7)–(4.8) on the manifold can be adapted from [14] and [15] and dropping terms of order $O(||B||_2^2)$ and $O(|u|^2)$ in the center-unstable manifold dynamics. The explicit form in terms of the adjoint impulse-free centre-unstable eigenspace follows the same reasoning as the proof of Proposition 3.1 from [9]. To get the final result 3), the linear-order in u dynamics on the center-unstable manifold can be computed similarly to in [9]:

$$\dot{u} = \Lambda u + O(||B||_{\mathbf{B}}^2)u, \qquad t \neq t_k \tag{4.14}$$

$$\Delta u = \Gamma\Psi(0)B_k\Phi_0 u(t^-) + O(||B||_{\mathbf{B}}^2)u(t^-), \qquad t = t_k. \tag{4.15}$$

Solutions of the above impulsive differential equation satisfy for $t \geq s$

$$u(t) = U(t,s)u_0 + \int_{t_0}^{t} U(t,s)O(||B||_{\mathbf{B}}^2)u(s)ds + \sum_{0<t_k\leq t} O(||B||_{\mathbf{B}}^2)u(t_k^-)$$

for U the fundamental matrix of (4.7)–(4.8) and $u_0 = u(s)$. Since r is a continuous growth rate of the latter, $||U(t,v)|| \leq Ke^{r(t-v)}$ for some $K > 0$. One can then use a Gronwall inequality argument to prove that

$$|u(t)| \leq K|u_0|e^{(t-t_0)(r+KO(||B||_{\mathbf{B}}^2))}\prod_{s<t_k\leq t}(1+KO(||B||_{\mathbf{B}}^2)) \leq K|u_0|e^{(r+O(||B||_{\mathbf{B}}|^2))(t-s)},$$

where to obtain the final bound we used the condition $t_{k+1} - t_k \geq p > 0$. The conclusion now follows from the attractivity of the center-unstable manifold; see the related result for the center manifold at Lemma 5.8.1 and Theorem 5.8.1 at [14]. □

Remark 9. The matrix $\Psi(t)$ is uniquely determined by the values of $\Psi(0)$, since $\Psi(t) = e^{-\Lambda t}\Psi(0)$. See Theorem 4.2, Chapter 7 from [11]. The requirements concerning the lower bound on time between impulses and the inequalities $t_k - r_j \neq t_m$ are technical; see Theorem 6.1.1 of [13].

Theorem 8 implies the following stabilization result, which provides an answer to our problem from Section II.

Corollary 10. *Choose some number ℓ of impulsive delays $r_1 < \cdots < r_\ell$. Let the impulse times t_k satisfy $t_{k+1} - t_k \geq p$ for some $p > 0$. Let the impulsive delays satisfy $t_k - r_j \neq t_m$ whenever $k > m \geq 0$ for all $j = 1, \ldots, \ell$. There exists $\delta > 0$ such that if the designed sequences of impulsive matrices $B = \{B_{0,k}, \ldots, B_{\ell,k}\}_{k\in\mathbb{N}}$ satisfies $||B||_{\mathbf{B}} < \delta$, then any continuous growth rate of the finite-dimensional system* (4.7)–(4.8) *is also a continuous growth rate for* (2.2)–(2.3) *up to a perturbation of size $O(||B||^2_{\mathbf{B}})$.*

V. Sufficient conditions for Periodic impulses

We will at this point work with a special class of controllers: periodic ones. The sequence of impulses is set to $t_k = \frac{k}{h}$ and the impulse matrices are constant with respect to k:

$$\dot{y} = A_0 y(t) + \sum_{j=1}^{m} A_j y(t - \tau_j), \qquad t \neq \frac{k}{h} \tag{5.16}$$

$$\Delta y = B_0 y(t^-) + \sum_{j=1}^{\ell} B_j y(t - r_j), \qquad t = \frac{k}{h}, \tag{5.17}$$

To continue we must allow complex-valued solutions. This is done in an analogous way to Definition 2.

Definition 11. A complex number μ is a *Floquet multiplier* of (5.16)–(5.17) if there exists a solution $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$ that satisfies the equation $x_{1/h} = \mu x_0$. A Floquet multiplier μ^* is *dominant* if all other Floquet multipliers μ satisfy $|\mu| \leq |\mu^*|$.

Proposition 12. *[Theorem 3.4.1 & Theorem 3.5.1, [14]] The linear system* (5.16)–(5.17) *is exponentially stable if and only if all of its Floquet multipliers μ satisfy $|\mu| < 1$.*

The following is a specification of Corollary 10 to the case of periodic impulses.

Theorem 13. *Suppose none of the impulsive delays r_j is an integer multiple of $\frac{1}{h}$. There exists $\delta > 0$ such that if $B = (B_0, \ldots, B_\ell)$ fulfills $||B|| < \delta$, the Floquet multipliers μ of* (5.16)–(5.17) *satisfying $|\mu| \geq 1$ correspond to those eigenvalues ζ satisfying $|\zeta| \geq 1 - \epsilon$ of the matrix*

$$\mathcal{M}_h(B) = \left(I + \Gamma\Psi(0) \left(B_0\Phi(0) + \sum_{k=1}^{\ell} B_k \Phi(-r_k) \right) \right) e^{\frac{1}{h}\Lambda}, \tag{5.18}$$

for some $\epsilon > 0$ small. Specifically, $\mu = \zeta \pm O(||B||^2_{\mathbf{B}})$.

Proof. For periodic impulses, (4.7)–(4.8) becomes

$$\dot{u} = \Lambda u, \qquad t \neq \frac{k}{h}$$

$$\Delta u = \Gamma\Psi(0) \left(B_0\Phi_0 + \sum_{j=1}^{\ell} B_j \Phi(-r_j) \right) u(t^-), \quad t = \frac{k}{h}.$$

The result follows from the theory of ordinary impulsive differential equations without delays [16] and Theorem 8 applied to (5.16)–(5.17). □

The following corollary can be proven much the same way as an analogous result from [9], so the proof will be omitted.

Corollary 14. *Suppose the impulse-free centre-unstable subspace has dimension q, none of the impulsive delays r_j is an integer multiple of $\frac{1}{h}$, and one of the following conditions is verified.*

1) *The image of the affine map $\mathcal{M}_h : (\mathbb{R}^{n\times n})^{\ell+1} \to \mathbb{R}^{q\times q}$ from* (5.18) *is the entirety of $\mathbb{R}^{q\times q}$.*
2) *$\Phi(0)$ and $\Psi(0)$ are rank q.*
3) *$A_1 = A_2 = \cdots = A_m = 0$.*
4) *$q = 1$.*

If the dominant Floquet multiplier of (2.1) *(interpreted as a trivially impulsive system with period $1/h$) is λ_0, there exists $\epsilon^* > 0$ such that for all $\epsilon \in [0, \epsilon^*)$, there exists $B(\epsilon) = (B_0(\epsilon), \ldots, B_\ell(\epsilon))$ such that dominant Floquet multiplier of* (5.16)–(5.17) *with $B = B(\epsilon)$ is $\lambda_0 - \epsilon$.*

The main point of Corollary 14 is that, under certain conditions, we are rigorously guaranteed that the plant (2.1) can at least be *partially stabilized* by periodic impulses, in the sense that its Floquet multipliers can be decreased.

Remark 15. Robustness of the conclusions of the above theorem and corollary with respect to non-periodic perturbations of the impulses is addressed by [17].

VI. A converse result

Corollary 14 gives a sufficient condition for partial stabilization. Here, we develop a converse.

Definition 16. Let all eigenvalues of the linear plant (2.1) have real part at most $\lambda_0 \geq 0$. This system is *infinitesimally stabilizable by impulses (ISI) with delay $\vec{r} = (r_1, \ldots, r_\ell)$ and frequency h* if there exists $\epsilon^* > 0$ and a differentiable function $[0, \epsilon^*) \ni \epsilon \mapsto B(\epsilon) \in (\mathbb{R}^{n\times n})^{\ell+1}$ with $B(0) = 0$ such that dominant Floquet multiplier of (5.16)–(5.17) with $B = B(\epsilon)$ is a strictly decreasing function of ϵ.

Corollary 17. *Suppose none of the impulsive delays r_j is an integer multiple of $\frac{1}{h}$, and let all nonnegative eigenvalues of* (2.1) *be simple. The plant is ISI with delay $\vec{r}$ and frequency h if and only if the restriction of the spectral radius function to* $\mathrm{Im}(\mathcal{M}_h)$ *possesses a smooth descent direction at $e^{\frac{1}{h}\Lambda}$ – that is, there exists $M \in \mathrm{Im}(\mathcal{M}_h)$ with $\rho(M) < e^{\lambda_0/h}$ and a differentiable homotopy $H : [0,1] \to \mathrm{Im}(\mathcal{M}_h)$ from $e^{\frac{1}{h}\Lambda}$ to M such that $x \mapsto \rho(H(x))$ is strictly decreasing.*

Proof. The forward direction is proven in a similar way to Corollary 14. To prove the converse, take the quotient $Q = (\mathcal{R}^{n\times n})^{\ell+1}/\ker(L)$, for

$$L(B) = \Gamma\Psi(0)B\Phi_0.$$

The induced map $\mathcal{M}_h/Q : Q \to \mathbb{R}^{q\times q}$ is well-defined and is an isomorphism onto its image. The homotopy H then induces a homotopy $\tilde{H} = (\mathcal{M}_h/Q)^{-1}H$ in Q. The latter can be identified with a homotopy in $(\mathbb{R}^{n\times n})^{\ell+1}$ by identifying Q with the orthogonal complement $\ker(L)^\perp$. By definition, $\tilde{H}(0) = (\mathcal{M}_h/Q)^{-1}(e^{\frac{1}{h}\Lambda}) = 0$. Let $\lambda_j(B)$ be the eigenvalue of $\mathcal{M}_h(B)$ with largest absolute value. From the assumptions

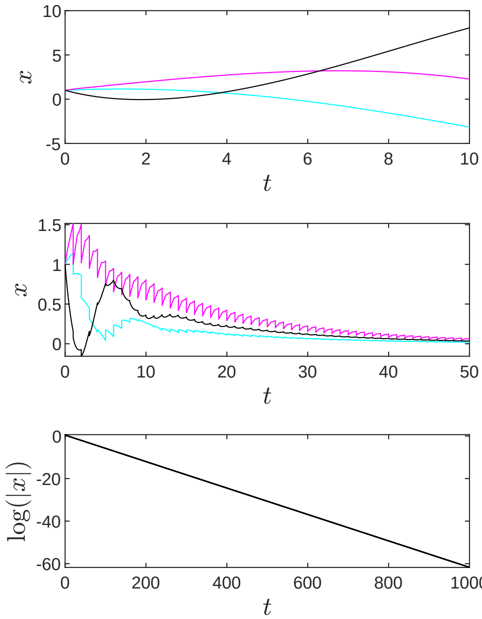

Fig. 7.1. Simulations of the system from Example VII from the constant initial condition $x_0 = (1,1,1)$. Blue, magenta and black denote the x_1, x_2 and x_3 components, respectively. Top: without impulses. Middle: with impulses. Bottom: log-scale plot of norm of the solution with impulses.

of the corollary, $x \mapsto \rho_j(x) \equiv |\lambda_j(\tilde{H}(x))|$ is decreasing and differentiable. Referring back to (4.14)–(4.15), let $\tilde{\lambda}_j(x)$ be the eigenvalue with $B = \tilde{H}(x)$ that coincides at $x = 0$ with $\lambda_j(0)$. Then $\tilde{\lambda}_j(x)$ is an eigenvalue of $\mathcal{M}_h(\tilde{H}(x)) + R(x)$, for a matrix $R(x)$ satisfying $||R(x)|| = O(|x|^2)$. It follows that $\frac{d}{dx}\tilde{\rho}_j(0) = \frac{d}{dx}\rho_j(0) < 0$, for $\tilde{\rho}_j(x) = |\tilde{\lambda}_j(x)|$. We conclude that $x \mapsto \tilde{\rho}_j(x)$ is strictly decreasing in a half neighbourhood of $x = 0$, say $[0, \epsilon_0)$. The result follows by taking $\epsilon^* \le \epsilon_0$ small enough so that $||\tilde{H}(\epsilon)|| < \delta$ for $\epsilon \in [0, \epsilon^*)$. □

VII. Example: Three-dimensional system

Consider the linear delay differential equation

$$\dot{x} = \begin{bmatrix} 0.3748 & -0.2571 & 0.1185 \\ 0.6150 & -1.0159 & 0.1335 \\ -1.4652 & 0.8497 & -0.6589 \end{bmatrix} x + \begin{bmatrix} 0 \\ x_2\left(t - \frac{1}{2}\right) \\ 0 \end{bmatrix}.$$

This system is unstable. We used the ISIM1s [18] package – an implementation of the method from Section V – to generate the impulsive controller

$$\Delta x = \begin{bmatrix} -0.1694 & -0.0786 & 0.2939 \\ -0.0787 & -0.2770 & -0.0300 \\ -0.0328 & -0.0348 & 0.0433 \end{bmatrix} x(t^-), \qquad t \in \frac{1}{h}\mathbb{Z}$$

with $h = 1$. We set a target to lower absolute values of Floquet multipliers to $e^{-0.1} < 1$. On the center-unstable manifold, this implies a continuous growth rate of -0.1. The result is a system that is asymptotically stable. See Figure 7.1. From the log scale plot we get an approximate continuous growth rate of $\lambda = -0.062$.

VIII. Example: Coupled delayed negative feedback network with ten nodes

Consider the ring network with delayed negative feedback:

$$\dot{x}_i = -\left(\frac{\pi}{2} + p_i\right) x_i(t-1) - 2x_i(t) + x_{i+1}(t) + x_{i-1}(t), \tag{8.19}$$

for indices $i = 0, \dots, 9$ being taken modulo 10, and $p_i \in [-1, 1]$. For this example, we had $\vec{p} = (p_0, \dots, p_9)$ with

$$\vec{p} = (-0.4953, 0.3595, 0.0141, 0.5567, -0.8785, 0.0354, -0.0800, 0.5579, 0.8081, 0.8797),$$

so considered independently (i.e. with coupling removed), seven nodes are unstable while three are stable. On the whole, this network is unstable.

We search for a matrix B such that (8.19) appended

$$\Delta x = Bx(t^-), \qquad t \in \frac{4}{5}\mathbb{Z} \tag{8.20}$$

is asymptotically stable. That is, $h = 5/4$. At this stage we do not impose any constraints on the structure of the matrix B. With ISIM1s set to a Floquet multiplier target of $e^{-0.08} < 1$, it generated the matrix in (8.6). This choice corresponds to a continuous growth rate of -0.1. Simulating the system with impulses, the result was stabilization; see the time-series in Figure 8.2.

As a secondary problem, we also searched for a matrix C with zero column sum such that (8.20) with B replaced with C stabilized (8.19). This ensures that the quantity $\sum_{i=1}^{10} x_i(t)$ is conserved at each impulse time t. For an additional constraint, we also required the matrix to have negative diagonal entries. The result of running ISIM1s with the same growth rate target was (8.7).

IX. Conclusion

We have presented an invariant manifold-guided impulsive stabilization framework for delay differential equations. Using centre-unstable subspace data for the system without impulses, the dynamics equation (4.7)–(4.8) on the parameter-dependent centre-unstable manifold for the impulsive delay equation can be used to derive a controller that lowers the growth/decay rate of solutions (Theorem 8, Corollary 10). The latter is a finite-dimensional impulsive system, for which the extant control theory can be applied. For periodic impulses, we obtain sufficient conditions under which the absolute value of the dominant Floquet multiplier can be lowered by a theoretically guaranteed amount (Corollary 14). One strength of this invariance-based stabilization idea is we are able to provide a partial answer to the question of when it is possible to stabilize (i.e. necessary and sufficient conditions) a delay equation by impulses (Corollary 17). As an application, we stabilized a three-dimensional system with a single time delay, and a ring network of ten nodes with delayed negative feedback.

$$B = \begin{bmatrix} -0.0722 & -0.0840 & -0.0787 & -0.0614 & -0.0199 & -0.0239 & -0.0340 & -0.0594 & -0.0749 & -0.0803 \\ -0.0764 & -0.1474 & -0.1557 & -0.1331 & -0.0251 & -0.0098 & 0.0056 & 0.0126 & 0.0038 & -0.0353 \\ -0.0701 & -0.1541 & -0.1712 & -0.1487 & -0.0345 & -0.0091 & 0.0201 & 0.0427 & 0.0384 & -0.0095 \\ -0.0535 & -0.1312 & -0.1485 & -0.1308 & -0.0305 & -0.0065 & 0.0233 & 0.0533 & 0.0553 & 0.0099 \\ -0.0242 & -0.0332 & -0.0430 & -0.0372 & -0.0339 & -0.0250 & -0.0107 & 0.0052 & 0.0124 & 0.0037 \\ -0.0296 & -0.0218 & -0.0219 & -0.0176 & -0.0265 & -0.0303 & -0.0307 & -0.0255 & -0.0176 & -0.0162 \\ -0.0406 & -0.0091 & 0.0048 & 0.0097 & -0.0126 & -0.0322 & -0.0552 & -0.0783 & -0.0798 & -0.0624 \\ -0.0669 & -0.0012 & 0.0309 & 0.0425 & 0.0064 & -0.0276 & -0.0825 & -0.1719 & -0.2094 & -0.1657 \\ -0.0812 & -0.0060 & 0.0326 & 0.0497 & 0.0168 & -0.0188 & -0.0861 & -0.2135 & -0.2761 & -0.2224 \\ -0.0825 & -0.0417 & -0.0129 & 0.0065 & 0.0098 & -0.0140 & -0.0639 & -0.1659 & -0.2203 & -0.1876 \end{bmatrix} \tag{8.6}$$

$$C = \begin{bmatrix} -0.00644 & -0.01879 & -0.01069 & -0.01274 & 0.01077 & -0.00269 & -0.01114 & 0.00716 & 0.02575 & 0.01732 \\ -0.01629 & -0.12547 & -0.12078 & -0.11630 & 0.04271 & 0.03835 & 0.04409 & 0.10016 & 0.12948 & 0.06863 \\ 0.02470 & -0.10111 & -0.13466 & -0.12974 & 0.00590 & 0.05218 & 0.11127 & 0.18377 & 0.19654 & 0.12474 \\ 0.02111 & -0.09452 & -0.12103 & -0.11868 & 0.00815 & 0.04244 & 0.09184 & 0.16901 & 0.19255 & 0.12516 \\ 0.08559 & 0.09231 & 0.02951 & 0.01880 & -0.08170 & -0.01062 & 0.07748 & 0.13860 & 0.14615 & 0.13998 \\ 0.04020 & 0.07308 & 0.05965 & 0.04341 & -0.04035 & -0.04020 & -0.01947 & 0.04277 & 0.09104 & 0.09820 \\ -0.01544 & 0.04899 & 0.08673 & 0.07071 & 0.00683 & -0.05957 & -0.11176 & -0.08247 & -0.01940 & 0.00847 \\ -0.04496 & 0.05743 & 0.10126 & 0.10171 & 0.01771 & -0.03671 & -0.10890 & -0.19377 & -0.21393 & -0.15165 \\ -0.04542 & 0.05960 & 0.08296 & 0.09912 & 0.00927 & 0.00003 & -0.05162 & -0.21629 & -0.31660 & -0.24227 \\ -0.04304 & 0.00849 & 0.02705 & 0.04370 & 0.02072 & 0.01679 & -0.02179 & -0.14896 & -0.23160 & -0.18859 \end{bmatrix} \tag{8.7}$$

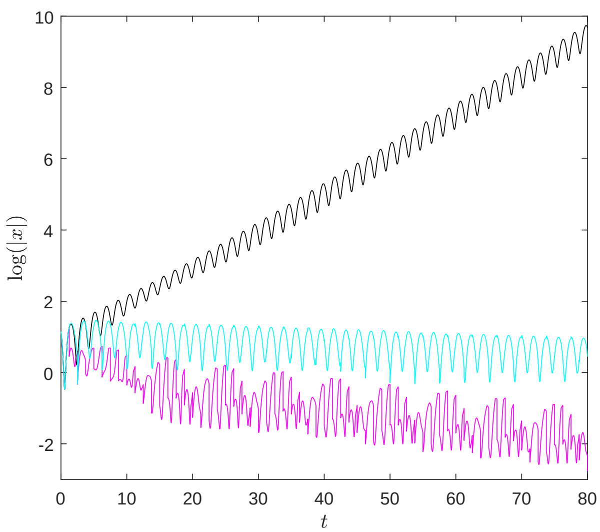

Fig. 8.2. Black: Solution of (8.19) without impulses. Magenta: With impulses at frequency $h = 4/5$ and the matrix $\tilde{B}$ in (8.6). Blue: with the matrix C in (8.7). All plots are from the constant initial condition $x_i = 1$ for $i = 1, \ldots, 10$ and plotted in logarithmic scale to emphasize stability/instability.

Acknowledgment

We thank the anonymous referees for their helpful comments and suggestions which led to several improvements to this note. K.E.M. Church acknowledges the support of NSERC (Natural Sciences and Engineering Research Council of Canada) through their Postdoctoral Fellowships Program.

References

[1] H. Wang, S. Duan, C. Li, L. Wang, and T. Huang, "Globally exponential stability of delayed impulsive functional differential systems with impulse time windows," *Nonlinear Dynamics*, vol. 84, no. 3, pp. 1655–1665, 2016.

[2] W. He, F. Qian, and J. Cao, "Pinning-controlled synchronization of delayed neural networks with distributed-delay coupling via impulsive control," *Neural Networks*, vol. 85, pp. 1–9, 2017. [Online]. Available: http://dx.doi.org/10.1016/j.neunet.2016.09.002

[3] M. S. Alwan and X. Liu, *Theory of Hybrid Systems: Deterministic and Stochastic*, ser. Nonlinear Physical Science. Singapore: Springer Singapore, 2018. [Online]. Available: http://link.springer.com/10.1007/978-981-10-8046-3

[4] I. M. Stamova and G. T. Stamov, "LyapunovRazumikhin method for impulsive functional differential equations and applications to the population dynamics," *Journal of Computational and Applied Mathematics*, vol. 130, no. 1-2, pp. 163–171, may 2001. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0377042799003854

[5] X. Yang, X. Li, Q. Xi, and P. Duan, "Review of stability and stabilization for impulsive delayed systems," *Mathematical Biosciences & Engineering*, vol. 15, no. 6, pp. 1495–1515, 2018. [Online]. Available: http://aimsciences.org//article/doi/10.3934/mbe.2018069

[6] T. Zhan, S. Ma, and H. Chen, "Impulsive stabilization of nonlinear singular switched systems with all unstable-mode subsystems," *Applied Mathematics and Computation*, vol. 344-345, pp. 57–67, mar 2019. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0096300318308695

[7] J. Lu, Z. Wang, J. Cao, D. W. C. Ho, and J. Kurths, "Pinning impulsive stabilization of nonlinear dynamical networks with time-varying delay," *International Journal of Bifurcation and Chaos*, vol. 22, no. 07, p. 1250176, 2012. [Online]. Available: http://www.worldscientific.com/doi/abs/10.1142/S0218127412501763

[8] X. Li and S. Song, "Stabilization of Delay Systems: Delay-Dependent Impulsive Control," *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 406–411, 2017.

[9] K. E. M. Church and X. Liu, "Cost-Effective Robust Stabilization and Bifurcation Suppression," *SIAM Journal on Control and Optimization*, vol. 57, no. 3, pp. 2240–2268, jan 2019. [Online]. Available: https://epubs.siam.org/doi/10.1137/18M1213142

[10] O. Diekmann, S. M. Verduyn Lunel, S. A. van Gils, and H.-O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, ser. Applied Mathematical Sciences. New York, NY: Springer New York, 1995, vol. 110. [Online]. Available: http://link.springer.com/10.1007/978-1-4612-4206-2

[11] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, ser. Applied Mathematical Sciences. New York, NY: Springer New York, 1993, vol. 99. [Online]. Available: http://link.springer.com/10.1007/978-1-4612-4342-7

[12] R. Fry and S. McManus, "Smooth bump functions and the geometry of banach spaces," *Expositiones Mathematicae*, vol. 20, no. 2, pp. 143–183, 2002.

[13] K. E. M. Church and X. Liu, "Smooth centre manifolds for impulsive delay differential equations," *Journal of Differential Equations*, vol. 265, no. 4, pp. 1696–1759, 2018. [Online]. Available: http://linkinghub.elsevier.com/retrieve/pii/S0022039618302122https://linkinghub.elsevier.com/retrieve/pii/S0022039618302122

[14] K. Church, "Invariant manifold theory for impulsive functional differential equations with applications," Ph.D. dissertation, University of Waterloo, 2019. [Online]. Available: http://hdl.handle.net/10012/14786

[15] K. E. M. Church and X. Liu, "Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations," *Journal of Differential Equations*, vol. 267, no. 6, pp. 3852–3921, sep 2019. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0022039619301810https://linkinghub.elsevier.com/retrieve/pii/S0022039619301810

[16] D. Bainov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*. Chapman and Hall/CRC, 1993.

[17] E. M. Bonotto, M. Federson, and F. L. Santos, "Robustness of Exponential Dichotomies for Generalized Ordinary Differential Equations," *Journal of Dynamics and Differential Equations*, 2019. [Online]. Available: https://doi.org/10.1007/s10884-019-09801-x

[18] K. E. M. Church, "User manual and tutorial for ISIM1s: a tiny MATLAB package for single stage invariant manifold-guided impulsive stabilization of delay equations," dec 2019. [Online]. Available: http://arxiv.org/abs/1912.07766